

University of Groningen

H ∞ Static Output-Feedback Gain-Scheduled Control for Discrete LPV Time-Delay Systems

Rosa, Tábita E.; Frezzatto, Luciano; Morais, Cecília F.; Oliveira, Ricardo C.L.F.

Published in:
Linear Parameter Varying Systems LPVS 2018

DOI:
[10.1016/j.ifacol.2018.11.155](https://doi.org/10.1016/j.ifacol.2018.11.155)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Final author's version (accepted by publisher, after peer review)

Publication date:
2018

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Rosa, T. E., Frezzatto, L., Morais, C. F., & Oliveira, R. C. L. F. (2018). H ∞ Static Output-Feedback Gain-Scheduled Control for Discrete LPV Time-Delay Systems. In A. Bazanella (Ed.), *Linear Parameter Varying Systems LPVS 2018* (pp. 137-142). (IFAC - PapersOnLine; Vol. 51, No. 26). IFAC.
<https://doi.org/10.1016/j.ifacol.2018.11.155>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

\mathcal{H}_∞ Static Output-Feedback Gain-Scheduled Control for Discrete LPV Time-Delay Systems^{*}

Tábitha E. Rosa^{*} Luciano Frezzatto^{**} Cecília F. Morais^{***}
Ricardo C. L. F. Oliveira^{***}

^{*} Faculty of Science and Engineering, University of Groningen, Groningen, The Netherlands, (e-mail: t.esteves.rosa@rug.nl).

^{**} Department of Electronics Engineering, Federal University of Minas Gerais, Belo Horizonte, MG, Brazil. (e-mail: lfrezzatto@cpdee.ufmg.br)

^{***} School of Electrical and Computer Engineering, University of Campinas – UNICAMP, Campinas, SP, Brazil. ({cfmorais, ricfow}@dt.fee.unicamp.br).

Abstract: This paper proposes new synthesis conditions to design \mathcal{H}_∞ static output-feedback controllers for discrete-time linear systems affected by time-varying parameters and time-varying delays. The design conditions are provided in terms of sufficient parameter-dependent linear matrix inequalities with a scalar parameter, being capable of synthesizing either robust or gain-scheduled controllers. The main motivations to deal with such problem are that many real-world plants can be modeled in terms of discrete-time linear parameter-varying (LPV) time-delay models and the lack of methods to deal with such systems considering an output-feedback based approach. The technique presented in this paper is quite generalist, allowing an arbitrary structure for the measured output matrix. Numerical examples are provided to illustrate the effectiveness of the synthesis conditions, tractable in terms of LMI relaxations, for robust or gain-scheduled \mathcal{H}_∞ output-feedback for LPV time-delayed systems.

Keywords: LPV systems, Time-delay systems, Discrete-time systems, \mathcal{H}_∞ control, Output-feedback, Gain-scheduling.

1. INTRODUCTION

Several real-world control applications deal with dynamics affected by an aftereffect phenomenon, also called *time-delay*. Physical processes found in biology, chemistry, epidemiology and engineering sciences, such as networked control systems and mechanical applications, can be described in terms of models with delayed structures (Richard, 2003). The presence of delays can be harmful to stability and performance of such systems, occasionally leading to unexpected oscillations, performance degradation and, in the worst case scenario, instability. Additionally, there are different classes of delays, each one influencing the behavior of the system in a different way. For instance, between time-varying and constant delays, the former is considered more prejudicial to the system stability than the latter. Moreover, even though there are transformations to cast discrete-time systems with constant delays into delay-free systems, those approaches cannot be directly applied to systems affected by time-varying delays, making these systems difficult to treat (Hu and Yuan, 2009). Another issue to take into account when dealing with time-delay systems is the rate of variation of the delay. As shown in Verriest (2010), the faster the delay varies, the greater is the damage to the system, which can lead, e.g., to the loss of causality.

Conventional techniques to handle the problems of stability analysis and stabilization of time-delay systems are, in general, based on the use of Lyapunov-Krasovskii functionals (Fridman,

2014; Briat, 2015), providing conservative analysis and synthesis conditions based on linear matrix inequalities (LMIs). An alternative that stands out in this context, and considering time-varying delays, is lifting the time-delay system into a switched delay-free system. It has been demonstrated in Hetel et al. (2008) that there is an equivalence between Lyapunov functionals used to certificate the stability of switched discrete-time systems and general delay-dependent Lyapunov-Krasovskii functionals used to assert the stability of discrete time-delay systems. Regarding performance criterion, it is demonstrated that the transformation proposed in Hu and Yuan (2009) is also valid in the context of determining \mathcal{H}_∞ guaranteed costs for discrete time-delay systems.

As an additional difficulty, dynamical systems may also have parameters that vary in time. If the underlying system is linear, it is commonly referred as a *linear parameter-varying* (LPV) system. One can also describe non-linear structures in terms of an LPV representation by linearizing the system on several points of interest (Rugh and Shamma, 2000). The combination of LPV and time-delay systems can potentially be applied to many practical applications, for instance, milling processes, robust fueling strategies for a spark ignition engine and open flow canals (Zhang et al., 2002; Zope et al., 2010; Blesa et al., 2010). There are methods in the literature to handle the design of filters, state-feedback controllers, dynamic output-feedback controllers or anti-windup compensators for discrete LPV time-delay systems (Han et al., 2014; Wu et al., 2006; Zope et al., 2012; Souza et al., 2017). However, to the best of authors' knowledge, the problem of designing static output-

^{*} Supported by the Brazilian agencies CAPES, CNPq (Grant 408782/2017-0), and FAPESP (Grants 2014/22881-1, 2017/01771-1 and 2017/18785-5) and the STW project 15472 of the STW Smart Industry 2016 program.

feedback controllers for this class of systems has not been investigated so far.

Based on the aforementioned discussion, this paper proposes new synthesis conditions in terms of parameter-dependent LMIs with a scalar parameter to treat the problem of \mathcal{H}_∞ static output-feedback control of discrete LPV time-delay systems. The proposed design method has a generalist nature regarding its application, being able to provide robust and mode-dependent gain-scheduled controllers considering either the output- or state-feedback problems. Additionally, differently from the techniques in the literature, the method also allows the treatment of measured-output and feed-forward matrices with arbitrary structures (uncertain or time-varying). Numerical examples are given to illustrate the effectiveness of the proposed method.

Notation: The set of natural numbers is denoted by \mathbb{N} , the set of real vectors (matrices) of order n ($n \times m$) is represented by \mathbb{R}^n ($\mathbb{R}^{n \times m}$), and the set of symmetric positive definite real matrices of order n is given by \mathbb{S}_+^n . For matrices or vectors, the symbol $'$ denotes the transpose, the expression $\text{He}(X) := X + X'$ is used to shorten formulas, the symbol \star represents transposed blocks in a symmetric matrix. To state that a symmetric matrix P is positive (negative) definite, it is used $P > 0$ ($P < 0$). The space of discrete functions that are square-summable is defined by ℓ_2 .

2. PROBLEM STATEMENT

Consider the following linear discrete-time system affected by time-varying parameters and time-varying delays

$$\begin{aligned} x(k+1) &= A(\alpha(k))x(k) + A_d(\alpha(k))x(k-\tau(k)) \\ &\quad + B(\alpha(k))u(k) + E(\alpha(k))w(k) \\ z(k) &= C_z(\alpha(k))x(k) + C_{zd}(\alpha(k))x(k-\tau(k)) \\ &\quad + D_z(\alpha(k))u(k) + E_z(\alpha(k))w(k) \\ y(k) &= C_y(\alpha(k))x(k) + C_{yd}(\alpha(k))x(k-\tau(k)) \\ &\quad + E_y(\alpha(k))w(k) \\ x(k) &= \phi(k), \forall k \in [-\bar{\tau}, 0] \end{aligned} \quad (1)$$

where $x(k) \in \mathbb{R}^{n_x}$ represents the state vector at the time $k \in \mathbb{N}$, $\tau(k) \in [\underline{\tau}, \bar{\tau}]$ is a positive integer representing the time-varying delay, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $w(k) \in \mathbb{R}^{n_w}$ is the exogenous input, $z(k) \in \mathbb{R}^{n_z}$ is the controlled output, $y(k) \in \mathbb{R}^{n_y}$ is the measured output, $\phi(k)$ is an initial condition sequence and $\alpha(k) = [\alpha_1(k), \dots, \alpha_N(k)]$ is a vector of bounded time-varying parameters, which lies in the unit simplex given by

$$\Lambda := \left\{ \zeta \in \mathbb{R}^N : \sum_{i=1}^N \zeta_i = 1, \zeta_i \geq 0, i = 1, \dots, N \right\},$$

for all $k \geq 0$. The state-space matrices of system (1) can be written as a convex combination of N known vertices as $M(\alpha(k)) = \sum_{i=1}^N \alpha_i(k)M_i$, $\alpha(k) \in \Lambda$.

Following the approach given in Hetel et al. (2008), consider the augmented state vector given by

$$\bar{x}(k) = [x'(k) \ x'(k-1) \ \dots \ x'(k-\bar{\tau})]'$$

Hence, system (1) can be reformulated as the following delay-free switched LPV system

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}^\kappa(\alpha(k))\bar{x}(k) + \bar{B}^\kappa(\alpha(k))u(k) + \bar{E}^\kappa(\alpha(k))w(k) \\ z(k) &= \bar{C}_z^\kappa(\alpha(k))\bar{x}(k) + D_z(\alpha(k))u(k) + E_z(\alpha(k))w(k) \\ y(k) &= \bar{C}_y^\kappa(\alpha(k))\bar{x}(k) + E_y(\alpha(k))w(k) \end{aligned} \quad (2)$$

where $\kappa \in \Omega := \{\underline{\tau}, \underline{\tau}+1, \dots, \bar{\tau}\}$ is a switching rule associated with the delay $\tau(k)$. Matrices $\bar{A}^\kappa(\alpha(k)) \in \mathbb{R}^{\bar{n}_x \times \bar{n}_x}$, $\bar{n}_x = (1 + \bar{\tau})n_x$, $\kappa \in \Omega$, are given by

$$\bar{A}^\kappa(\alpha(k)) = \begin{bmatrix} A(\alpha(k)) & A_d^\kappa(\alpha(k)) \\ \Phi_1 & \Phi_2 \end{bmatrix}$$

with $\Phi_1 = \text{diag}(\mathbf{I}, \mathbf{I}, \dots, \mathbf{I}) \in \mathbb{R}^{\bar{\tau}n_x \times \bar{\tau}n_x}$, $\Phi_2 = [0 \ 0 \ \dots \ 0]' \in \mathbb{R}^{\bar{\tau}n_x \times n_x}$ and

$$A_d^\kappa(\alpha(k)) = [0_{n_x \times (\tau(k)-1)n_x} \ A_d(\alpha(k)) \ 0_{n_x \times (\bar{\tau}-\tau(k))n_x}]$$

Matrices $\bar{B}^\kappa(\alpha(k))$, $\bar{E}_z^\kappa(\alpha(k))$, $\bar{C}_z^\kappa(\alpha(k))$, $\bar{C}_y^\kappa(\alpha(k))$, $\kappa \in \Omega$, are given by

$$[\bar{B}^\kappa(\alpha(k)) \ | \ \bar{E}^\kappa(\alpha(k))] = \begin{bmatrix} B(\alpha(k)) & E(\alpha(k)) \\ 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \bar{C}_z^\kappa(\alpha(k)) &= [C_z(\alpha(k)) \ C_z^\kappa(\alpha(k))] \\ \bar{C}_y^\kappa(\alpha(k)) &= [C_y(\alpha(k)) \ C_y^\kappa(\alpha(k))] \end{aligned}$$

where matrices $C_z^\kappa(\alpha(k))$ and $C_y^\kappa(\alpha(k))$ are constructed similarly to matrix $\bar{A}^\kappa(\alpha(k))$.

The purpose of this paper is to design a stabilizing mode-dependent (or delay-dependent) gain-scheduled static output-feedback control law given by $u(k) = \Theta^\kappa(\alpha(k))y(k)$, where $\Theta^\kappa(\alpha(k)) \in \mathbb{R}^{n_u \times n_y}$, assuring an \mathcal{H}_∞ guaranteed cost bounded by μ . Applying the proposed control law in system (1), one obtains the closed-loop system given by

$$\mathcal{H} := \begin{cases} \bar{x}(k+1) = A_{cl}^\kappa(\alpha(k))\bar{x}(k) + B_{cl}^\kappa(\alpha(k))w(k) \\ z(k) = C_{cl}^\kappa(\alpha(k))\bar{x}(k) + D_{cl}^\kappa(\alpha(k))w(k) \end{cases} \quad (3)$$

whose matrices are given by

$$\begin{aligned} \begin{bmatrix} A_{cl}^\kappa(\alpha(k)) & B_{cl}^\kappa(\alpha(k)) \\ C_{cl}^\kappa(\alpha(k)) & D_{cl}^\kappa(\alpha(k)) \end{bmatrix} &= \begin{bmatrix} \bar{A}^\kappa(\alpha(k)) & \bar{E}^\kappa(\alpha(k)) \\ \bar{C}_z^\kappa(\alpha(k)) & E_z(\alpha(k)) \end{bmatrix} \\ &+ \begin{bmatrix} \bar{B}^\kappa(\alpha(k)) \\ D_z(\alpha(k)) \end{bmatrix} \Theta(\alpha(k)) [\bar{C}_y^\kappa(\alpha(k)) \ E_y(\alpha(k))]. \end{aligned} \quad (4)$$

The \mathcal{H}_∞ norm is used to represent an optimization criterion associated to disturbance rejection and its upper bound μ can be computed, for instance, by taking the definition presented in De Caigny et al. (2010); Hu and Yuan (2009), which assures that, for any input $w(k) \in \ell_2$, the output of the system $z(k)$ satisfies

$$\|z(k)\|_2 < \mu \|w(k)\|_2, \mu > 0, \forall \alpha(k) \in \Lambda, k \geq 0, \forall \kappa \in \Omega.$$

3. MAIN RESULTS

This section presents sufficient parameter-dependent LMI conditions for the synthesis of \mathcal{H}_∞ static output-feedback gain-scheduled controllers for system (1), which is the main contribution of this paper.

Theorem 1. For a given scalar $\gamma \neq 0$ and a matrix $Q^\kappa(\alpha(k))$, if there exist matrices $P^\kappa(\alpha(k)) \in \mathbb{S}_+^{\bar{n}_x}$, $G^\kappa(\alpha(k)) \in \mathbb{R}^{\bar{n}_x \times \bar{n}_x}$, $L^\kappa(\alpha(k)) \in \mathbb{R}^{n_u \times n_y}$ and $S^\kappa(\alpha(k)) \in \mathbb{R}^{n_y \times n_y}$, and a scalar $\mu > 0$ such that inequality¹ (5) holds for all $(\alpha(k), \alpha(k+1)) \in \Lambda \times \Lambda$, and $\kappa, \iota \in \Omega$, then the stabilizing mode-dependent static output-feedback gain-scheduled controller given by

$$\Theta^\kappa(\alpha(k)) = L^\kappa(\alpha(k))S^\kappa(\alpha(k))^{-1}$$

assures the closed-loop asymptotic stability and also that μ is an \mathcal{H}_∞ guaranteed cost for system (3).

¹ For ease of notation, the dependence on $\alpha(k)$ is omitted in this inequality and in the proof of Theorem 1. Furthermore, P_+^ι is used to represent $P^\iota(\alpha(k+1))$.

$$\begin{bmatrix} -P_+^\iota & \star & \star & \star \\ (\bar{A}^\kappa G^\kappa + \bar{B}^\kappa L^\kappa Q^\kappa)' & -G^\kappa - \bar{G}^{\kappa'} + P^\kappa & \star & \star \\ 0 & \bar{C}_z^\kappa G^\kappa + D_z L^\kappa Q^\kappa & -\mu^2 \mathbf{I} & \star \\ (\bar{E}^\kappa + \bar{B}^\kappa L^\kappa E_y)' & 0 & (E_z + D_z L^\kappa E_y)' & -\mathbf{I} \\ (\bar{B}^\kappa L^\kappa)' & \gamma(S^\kappa Q^\kappa - \bar{C}_y^\kappa G^\kappa) & (D_z L^\kappa)' & \gamma(S^\kappa E_y - E_y) \quad \gamma(S^\kappa + S^{\kappa'})' \end{bmatrix} < 0 \quad (5)$$

Proof. First note that the feasibility of (5) guarantees that $\gamma(S^\kappa + S^{\kappa'}) < 0$, implying that the $S^{\kappa^{-1}}$ exists. Pre- and post-multiplying (5), respectively, by

$$\mathcal{B}^{\perp'} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} 0 \\ (S^{\kappa^{-1}} \bar{C}_y^\kappa G^\kappa - Q^\kappa)' \\ 0 \\ (S^{\kappa^{-1}} E_y^\kappa - E_y)' \end{bmatrix}$$

and \mathcal{B}^\perp , yields

$$\begin{bmatrix} -P_+^\iota & \star & \star & \star \\ 0 & P^\kappa & \star & \star \\ 0 & 0 & -\mu^2 \mathbf{I} & \star \\ B_{cl}^\kappa' & 0 & D_{cl}^\kappa & -\mathbf{I} \end{bmatrix} + \text{He} \left(\begin{bmatrix} A_{cl}^\kappa \\ -\mathbf{I} \\ C_{cl}^\kappa \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ G^{\kappa'} \\ 0 \\ 0 \end{bmatrix}' \right) < 0, \quad (6)$$

with A_{cl}^κ , B_{cl}^κ , C_{cl}^κ , and D_{cl}^κ as given in (4). The next step is to pre- and post-multiply (6) respectively by

$$\mathcal{R}' = \begin{bmatrix} \mathbf{I} & A_{cl}^\kappa & 0 & 0 \\ 0 & C_{cl}^\kappa & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{bmatrix},$$

and \mathcal{R} , resulting in

$$\begin{bmatrix} A_{cl}^\kappa P^\kappa A_{cl}^{\kappa'} - P_+^\iota & \star & \star \\ C_{cl}^\kappa P A_{cl}^{\kappa'} & -\mu^2 \mathbf{I} + C_{cl}^\kappa \hat{P} C_{cl}^{\kappa'} & \star \\ B_{cl}^\kappa' & D_{cl}^\kappa & -\mathbf{I} \end{bmatrix} < 0,$$

which can be recognized as the Bounded Real Lemma (de Souza et al., 2006, Lemma 3) applied to the switched LPV system (3), which guarantees the asymptotic stability and that μ is an upper bound for the \mathcal{H}_∞ norm of system (3).

Remark 1. Note that the technique employed to derive the conditions of Theorem 1 has some similarities with the one used in the so called *two-stages approach* (Peaucelle and Arzelier, 2001; Mehdi et al., 2004; Agulhari et al., 2010), where a stabilizing state-feedback gain must be computed in the first step. Both methods use the elimination lemma but Theorem 1 is solved in only one step.

The following corollary presents an adaptation of Theorem 1 to handle the stabilization of system (3) free of exogenous inputs ($w(k) = 0$).

Corollary 1. For a given scalar $\gamma \neq 0$ and a matrix $Q^\kappa(\alpha(k))$, if there exist matrices $P^\kappa(\alpha(k)) \in \mathbb{S}_+^{\bar{n}_x}$, $G^\kappa(\alpha(k)) \in \mathbb{R}^{\bar{n}_x \times \bar{n}_x}$, $L^\kappa(\alpha(k)) \in \mathbb{R}^{n_u \times n_y}$ and $S^\kappa(\alpha(k)) \in \mathbb{R}^{n_y \times n_y}$ such that inequality (5) without third and fourth rows and columns holds for all $(\alpha(k), \alpha(k+1)) \in \Lambda \times \Lambda$ and $\kappa, \iota \in \Omega$, then the stabilizing mode-dependent static output-feedback gain-scheduled controller given by $\Theta^\kappa(\alpha(k)) = L^\kappa(\alpha(k)) S^\kappa(\alpha(k))^{-1}$ assures that system (3) with $w(k) = 0$ is asymptotically stable.

Next remark shows some possibilities regarding the structure and requirements of the controllers provided by Theorem 1 and Corollary 1.

Remark 2. Note that the control gains provided by Theorem 1 and Corollary 1 are mode-dependent gain-scheduled. When the values of the time-delay or of the scheduling parameters are not available in real-time for feedback purposes (or there is

no interest in using these information), some other particular structures can be obtained by setting the matrices $S^\kappa(\alpha(k))$ and $L^\kappa(\alpha(k))$ as follows:

Mode-dependent: $S^\kappa(\alpha(k)) = S^\kappa$, $L^\kappa(\alpha(k)) = L^\kappa$.

Gain-scheduled: $S^\kappa(\alpha(k)) = S(\alpha(k))$, $L^\kappa(\alpha(k)) = L(\alpha(k))$.

Robust: $S^\kappa(\alpha(k)) = S$, $L^\kappa(\alpha(k)) = L$.

The last structure tends to provide the most conservative results but, on the other hand, requires the simplest and cheapest implementation.

Matrices $Q^\kappa(\alpha(k))$ are introduced in Theorem 1 in order to linearize the inequalities associated to the output-feedback problem (otherwise it would be necessary to deal with bilinear matrix inequalities – BMIs). Since the dimensions imposed to the matrix $Q^\kappa(\alpha(k))$ are equal to the dimensions of the measured output matrix $\bar{C}_y^\kappa(\alpha(k))$ from the augmented switched system, an intuitive choice is setting $Q^\kappa(\alpha(k)) = \bar{C}_y^\kappa(\alpha(k))$. Another possible choice is given by

$$Q^\kappa = [0_{n_y \times \sigma_Q} \quad \mathbf{I}_{n_y} \quad 0_{n_y \times (\bar{n}_x - \sigma_Q - n_y)}], \quad (7)$$

where a new input parameter, $0 \leq \sigma_Q \leq \bar{n}_x - n_y$, is introduced to define the position of the identity matrix in (7).

One particularity of the proposed technique is the possibility of performing searches on the scalar γ . This parameter needs to be chosen beforehand, otherwise (5) would be a BMI. Results with different levels of conservativeness are obtained by varying the values of this scalar. Further details about this subject are given in Section 4.

Theorem 1 and Corollary 1 can be straightforwardly extended to deal with other classes of dynamical systems besides discrete LPV time-delay systems. The first extension is the problem of state-feedback control, which can be achieved by replacing $\bar{C}_y^\kappa(\alpha(k))$ and $E_y(\alpha(k))$ by \mathbf{I} and 0 , respectively, in system (2). To treat linear time-invariant (LTI) systems, consider matrices of system (1) and the decision variables of Theorem 1 and Corollary 1 depending on time-invariant parameters ($\alpha(k) = \alpha, \forall k \in \mathbb{N}$), and set $P^\iota(\alpha(k+1)) = P^\iota(\alpha(k)) = P^\iota(\alpha)$.

The proposed method can also be employed to handle systems whose time-varying delays have bounded rates of variation. This case can be found, for instance, in physical processes where it is not reasonable to assume the delay varying from the minimum to the maximum value in only one instant of time. In discrete-time context, there are only a few methods in the literature that considers this approach, such as Silva et al. (2016); Souza et al. (2017). One can consider the variation of the delay in consecutive samples to be limited by $|\tau(k+1) - \tau(k)| \leq \Delta\tau_{\max} < \bar{\tau}$. Therefore, Theorem 1 and Corollary 1 can be rewritten considering $\kappa = \underline{\tau}, \dots, \bar{\tau}$ and $\iota = \max(\underline{\tau}, \tau - \Delta\tau_{\max}), \dots, \min(\bar{\tau}, \tau + \Delta\tau_{\max})$.

Another appealing feature of the proposed technique is the possibility of dealing with any output matrix $C_y(\alpha(k))$ without imposing special structures or constraints on the optimization variables, since for most of the methods found in the literature, this matrix is required to be constant, parameter-independent

and constrained to the form $C_y(\alpha(k)) = [I \ 0]$, or even to undergo similarity transformations (Peres et al., 1994; Dong and Yang, 2013).

4. FINITE DIMENSIONAL TESTS

This section presents a few considerations necessary to perform numerical tests using the proposed method. The first issue is the variation of parameter $\alpha(k)$ and two scenarios are possible in this context: bounded rate of variation ($\alpha(k+1)$ depends on $\alpha(k)$) and arbitrarily fast variation (both parameters are independent) (Oliveira and Peres, 2009). In this paper, the latter case is adopted in the numerical experiments and the following change of variables is used: $\alpha(k+1) = \beta(k) \in \Lambda$. Even after these considerations, the proposed conditions are not in a programmable form yet since they are given as parameter-dependent (robust) LMIs. To overcome this issue, it is employed the strategy proposed in Oliveira and Peres (2007), basically imposing polynomial structures to the decision variables and applying a relaxation, for instance, the Pólya's relaxation (Hardy et al., 1952), to check the positivity of the resulting polynomial matrix inequalities. The MATLAB parser ROLMIP (Robust LMI Parser) (Agulhari et al., 2012) may be used to automate this procedure. This parser is able to extract a finite set of LMIs from polynomial positivity tests after imposing a fixed degree for the decision variables.

The optimization variables can depend polynomially on the parameters with different degrees. The structure of the controller is defined by the variables $L^\kappa(\alpha(k))$ and $S^\kappa(\alpha(k))$, and if the desired controller is robust, then both matrices must have zero degree. A gain-scheduled controller is obtained if, at least, one of the degrees associated to either $L^\kappa(\alpha(k))$ or $S^\kappa(\alpha(k))$ is different than zero. In this case, the vector $\alpha(k)$ must be available on-line (measured or estimated). The other optimization variables can also depend on the parameters and the chosen degrees only affect the conservativeness of the solutions. As a general rule, higher degrees may produce improved solutions at the price of a larger computational effort. To perform the numerical examples of this paper, these variables are kept with degree equal to one.

As mentioned before, the proposed conditions require the parameter γ to be given *a priori*. In this paper it is not investigated how to perform the search in this scalar. Instead, a set of values given by

$$\gamma \in \{-1, -10^{-1}, -10^{-2}, -10^{-3}, -10^{-4}\} \quad (8)$$

is used in the numerical experiments of Section 5. Testing more values or performing a search based on some optimization method could improve the results at the price of a larger computational burden.

5. NUMERICAL EXAMPLES

All the conditions proposed in this paper were programmed using the software MATLAB (R2014a) with the aid of the parsers ROLMIP (Agulhari et al., 2012) and Yalmip (Löfberg, 2004) and of the solver Mosek (ApS, 2015).

Regarding the choices for matrices $Q^\kappa(\alpha(k))$, tests were made using $Q^\kappa(\alpha(k)) = \bar{C}_y^\kappa(\alpha(k))$ and also combining it with the structure outlined in (7). The performance obtained with the second choice is, however, slightly more conservative than the first. Therefore, all numerical results presented in this

paper were obtaining using matrices $Q^\kappa(\alpha(k))$ equals to the measured-output matrix.

Example 1 The system investigated in this example is given in Zhang et al. (2007), where the dynamic matrices originally represent a switched system with two second order subsystems besides a delayed dynamic matrix A_d . In this example the subsystems are considered as the vertices of an LPV system, that is, the system can vary inside the polytope formed by the two subsystems instead of only switching between them. Two scenarios are investigated. In the first one, the variation of the delay is considered to be arbitrary, while, in the second, it is considered that this rate is limited by $\Delta\tau_{\max} = 1$. The value of the delay is not available in real-time and, in this case, mode-independent controllers are the only choice.

Table 1 shows the \mathcal{H}_∞ guaranteed costs associated to the robust and the gain-scheduled static output-feedback controllers designed by Theorem 1, assuming different delay ranges for both approaches. The reported results correspond to the value of γ that provided the best (less conservative) upper bound to the \mathcal{H}_∞ norm of the closed-loop system (3), among the values given in (8). Note that, as expected, the gain-scheduled con-

Table 1. \mathcal{H}_∞ guaranteed costs associated to the static output-feedback controllers designed by Theorem 1 for Example 1.

| | $\underline{\tau}$ | $\overline{\tau}$ | Robust | | Gain-scheduled | |
|-----------|--------------------|-------------------|--------|----------|----------------|----------|
| | | | μ | γ | μ | γ |
| Arbitrary | 2 | 4 | 0.3437 | -0.1 | 0.3213 | -0.1 |
| | 2 | 5 | 0.3675 | -0.1 | 0.3439 | -0.1 |
| | 2 | 6 | 0.3948 | -0.1 | 0.3722 | -0.1 |
| | 2 | 7 | 0.4185 | -0.01 | 0.4078 | -0.1 |
| Limited | 2 | 4 | 0.3437 | -0.1 | 0.3213 | -0.1 |
| | 2 | 5 | 0.3588 | -0.1 | 0.3345 | -0.1 |
| | 2 | 6 | 0.3674 | -0.001 | 0.3470 | -0.1 |
| | 2 | 7 | 0.3613 | -0.1 | 0.3377 | -0.1 |

trollers provided improved performance when compared to the robust ones in both scenarios of delay variation. Additionally, observe that, as the range of delays increases, the \mathcal{H}_∞ guaranteed costs also increase, which is expected since the synthesis conditions must hold for a larger delay range. Furthermore, one may verify that when the delay variation is limited, both robust and gain-scheduled controllers provide better \mathcal{H}_∞ guaranteed costs than in the case of arbitrary variation.

Example 2 This example considers a non-linear system represented as a two-rule Takagi-Sugeno fuzzy model given in Dong et al. (2010). The system is affected by multiple communication delays and has multiple missing measurements. Similar to the procedure adopted in Example 1, the dynamic matrices of the fuzzy system are taken as the vertices of an LPV system, enabling the application of Theorem 1 to synthesize \mathcal{H}_∞ stabilizing static output-feedback controllers considering $\tau \in [2, 6]$ and $\Delta\tau_{\max} = 1$. The value of the delay as well as the scheduling parameters are not available in real-time, posing the most challenging design scenario, that is, the controller must be mode-independent and robust. Applying the conditions of Theorem 1, the following robust mode-independent gain (truncated with 4 decimal digits)

$$K = LS^{-1} = \begin{bmatrix} -0.0139 & 0.0256 \\ -0.0073 & -0.0074 \end{bmatrix} \quad (9)$$

is obtained ($\gamma = -1$, $\mu = 0.2022$). To show that the designed controller is stabilizing, time simulations were performed considering null initial conditions and considering the particular case where the uncertain parameter $\alpha(k) = (\alpha_1(k), \alpha_2(k))$ is given by $\alpha_1(k) = (\sin(4.56k) \cos(9.12k) + 1)/2$ and $\alpha_2(k) = 1 - \alpha_1(k)$, and the exogenous inputs are given by $w(k) = 10e^{-0.1(\tau(k) - \bar{\tau})} \sin(5(\tau(k) - \bar{\tau}) + \tilde{w}_k$, where \tilde{w}_k is a white Gaussian noise with null mean and covariance $\sigma^2 = 0.2$.

Fig. 1 presents a time-response of the controlled output $z(k)$ of the open-loop system, from which it is noticeable that the system has an unstable behavior. Fig. 2 presents the time-response of the controlled output $z(k)$ of the closed-loop system using the output-feedback controller (9) synthesized by Theorem 1. As expected, the performed simulations show that the output trajectories converge to zero (stable closed-loop system).

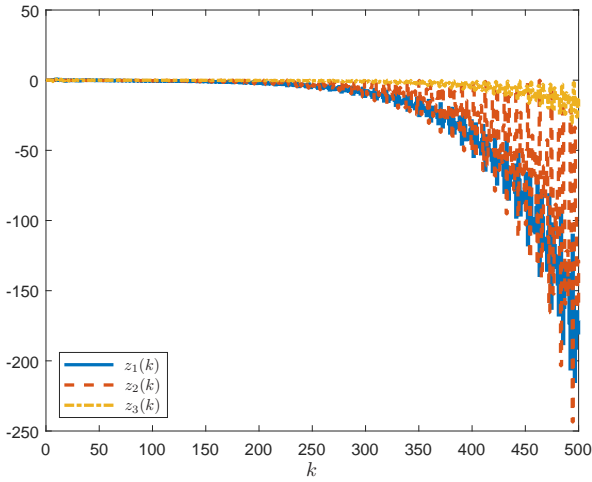


Fig. 1. Trajectories of the controlled output of the open-loop system of Example 2 with $\tau \in [2, 6]$ and the variation of the delay is $\Delta\tau_{\max} = 1$.

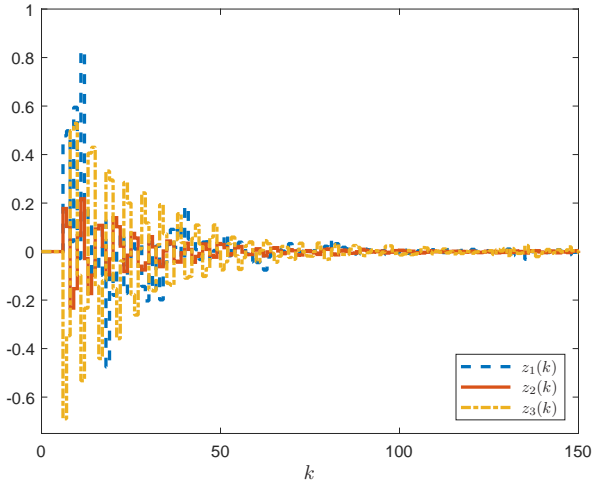


Fig. 2. Trajectories of the controlled output of the closed-loop system of Example 2 considering controllers (9) synthesized by Theorem 1 with $\tau \in [2, 6]$ and the variation of the delay is $\Delta\tau_{\max} = 1$.

Example 3 This example investigates the two-vertices uncertain time-invariant system presented in Caldeira et al. (2011) whose matrices of the first vertex are given by

$$A = \begin{bmatrix} -0.5 & 1 \\ 0 & 0.2 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.05 & 0.1 \\ 0 & 0.02 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix},$$

$E = [1 \ 0.3]'$, $C_z = [1 \ 3]$, $C_{zd} = [0 \ 0]$, $D_z = 1$, $E_z = 0$ and the matrices of the second vertex are obtained by multiplying the first ones by a scalar β . The matrices associated to the measured output are constant $C_y = [1 \ 3]$, $C_{yd} = [0 \ 0]$ and $E_y = 0$ (not affected by β).

The aim of this example is to compare the \mathcal{H}_∞ performance of the closed-loop LTI system considering robust static mode-independent (the value of the delay is not available) output-feedback controllers obtained by Theorem 1 and a condition adapted from Theorem 4 of Leite et al. (2011). Note that, in order to compute static output-feedback controllers using the technique from Leite et al. (2011), it is required that the output matrix C_y is constant and that a similarity transformation be applied to system (1) to ensure that $C_y = [I \ 0]$. The transformation used is $T^{-1} = [C_y' (C_y C_y')^{-1} C_y^\perp]$. It is important to mention that, even though the technique from Leite et al. (2011) requires a small computational effort (in terms of scalar decision variables and number of LMI rows) because it is based on a particular Lyapunov-Krasovskii function (and not based on the switched-system approach), the method proposed in this paper leads to less conservative results in terms of \mathcal{H}_∞ guaranteed costs. To illustrate the behavior of the methods, Fig. 3 presents the \mathcal{H}_∞ guaranteed costs obtained by the method from Leite et al. (2011) and Theorem 1 supposing an arbitrarily varying time-delay in the interval $[1, \bar{\tau}]$, $\bar{\tau} = \{2, 3, 4\}$, and considering the parameter $\beta \in [1 \ 1.1]$. From the figure, it is possible to note that Theorem 1 provides less conservative performance than the technique from Leite et al. (2011).

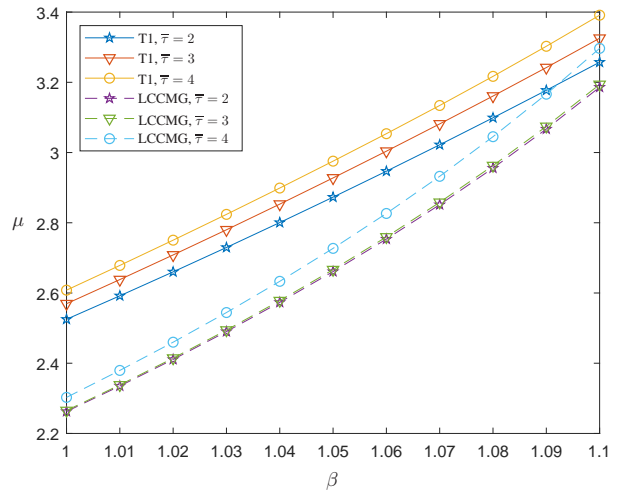


Fig. 3. \mathcal{H}_∞ guaranteed costs provided by Theorem 1 and Theorem 4 from Leite et al. (2011) for different ranges of delay for Example 3

6. CONCLUSION

This paper proposed new parameter-dependent LMI conditions for the synthesis of \mathcal{H}_∞ static output-feedback controllers for discrete LPV time-delay systems. One advantage of the proposed method is its versatility, being capable of designing static output- or state-feedback controllers for either time-delay or

delay-free LPV and LTI systems. A second important advantage is the possibility of considering a time-varying output matrix $C_y(\alpha(k))$ while other techniques in the literature require this matrix to be parameter-independent or to have a particular structure. This flexibility in the measured matrix can be useful, for example, to deal with networked systems in which this matrix is often uncertain. Numerical experiments based on LPV models borrowed from the literature demonstrated the applicability and flexibility of the approach, that can be less conservative for the design of controllers than some existing methods in terms of improved \mathcal{H}_∞ guaranteed costs. The next step of the research is to consider that not only the states but also the time-varying parameters are affected by delays.

REFERENCES

- Agulhari, C.M., Oliveira, R.C.L.F., and Peres, P.L.D. (2010). Static output feedback control of polytopic systems using polynomial Lyapunov functions. In *Proc. 49th IEEE Conf. Decision Control*, 6894–6901. Atlanta, GA, USA.
- Agulhari, C.M., Oliveira, R.C.L.F., and Peres, P.L.D. (2012). Robust LMI parser: A computational package to construct LMI conditions for uncertain systems. In *XIX CBA*, 2298–2305. Campina Grande, PB, Brasil. <http://rolmip.github.io/>
- ApS, M. (2015). *The MOSEK optimization software*. <http://www.mosek.com>.
- Blesa, J., Puig, V., and Bolea, Y. (2010). Fault detection using interval LPV models in an open-flow canal. *Control Eng. Pract.*, 18(5), 460–470.
- Briat, C. (2015). *Linear Parameter-Varying and Time-Delay Systems — Analysis, Observation, Filtering and Control*, volume 3 of *Advances in Delays and Dynamics*. Springer-Verlag, Berlin Heidelberg.
- Caldeira, A.F., Leite, V.J., Miranda, M.F., Castro, M.F., and Gonçalves, E.N. (2011). Convex robust \mathcal{H}_∞ control design to discrete-time systems with time-varying delay. *IFAC Proceedings Volumes*, 44(1), 10150–10155.
- De Caigny, J., Camino, J.F., Oliveira, R.C.L.F., Peres, P.L.D., and Swevers, J. (2010). Gain-scheduled \mathcal{H}_2 and \mathcal{H}_∞ control of discrete-time polytopic time-varying systems. *IET Control Theory & Appl.*, 4(3), 362–380.
- de Souza, C.E., Barbosa, K.A., and Trofino, A. (2006). Robust \mathcal{H}_∞ filtering for discrete-time linear systems with uncertain time-varying parameters. *IEEE Trans. Signal Process.*, 54(6), 2110–2118.
- Dong, H., Wang, Z., Ho, D.W., and Gao, H. (2010). Robust \mathcal{H}_∞ fuzzy output-feedback control with multiple probabilistic delays and multiple missing measurements. *IEEE Trans. Fuzzy Syst.*, 18(4), 712–725.
- Dong, J. and Yang, G.H. (2013). Robust static output feedback control synthesis for linear continuous systems with polytopic uncertainties. *Automatica*, 49(6), 1821–1829.
- Fridman, E. (2014). *Introduction to Time-Delay Systems: Analysis and Control*. Birkhäuser Basel, Switzerland.
- Han, C., Zhang, F., Liu, L., Bi, S., and Sun, D. (2014). Adaptive robust \mathcal{H}_∞ control for uncertain discrete-time systems with time-varying state and input delays. In *Proc. of 2014 Guidance, Navigation and Control Conf.*, 2236–2239. Yantai, China.
- Hardy, G.H., Littlewood, J.E., and Pólya, G. (1952). *Inequalities*. Cambridge University Press, Cambridge, UK, 2 edition.
- Hotel, L., Daafouz, J., and Iung, C. (2008). Equivalence between the Lyapunov–Krasovskii functionals approach for discrete delay systems and that of the stability conditions for switched systems. *Nonlinear Analysis: Hybrid Systems*, 2(3), 697–705.
- Hu, K. and Yuan, J. (2009). Delay-dependent \mathcal{H}_∞ control of linear discrete-time systems with time-varying delay via switched system approach. *Int. J. Adapt. Control Signal Process.*, 23(12), 1104–1112.
- Leite, V.J.S., Castro, M.F., Caldeira, A.F., Miranda, M.F., and Gonçalves, E.N. (2011). Uncertain discrete-time systems with delayed state: robust stabilization with performance specification via LMI formulations. In M.A. Jordán (ed.), *Discrete Time Systems*, chapter 17, 295–326. InTech.
- Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proc. 2004 IEEE Int. Symp. on Comput. Aided Control Syst. Des.*, 284–289. Taipei, Taiwan.
- Mehdi, D., Boukas, E.K., and Bachelier, O. (2004). Static output feedback design for uncertain linear discrete time systems. *IMA J. Math. Control Inform.*, 21(1), 1–13.
- Oliveira, R.C.L.F. and Peres, P.L.D. (2007). Parameter-dependent LMIs in robust analysis: Characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations. *IEEE Trans. Autom. Control*, 52(7), 1334–1340.
- Oliveira, R.C.L.F. and Peres, P.L.D. (2009). Time-varying discrete-time linear systems with bounded rates of variation: Stability analysis and control design. *Automatica*, 45(11), 2620–2626.
- Peaucelle, D. and Arzelier, D. (2001). Robust performance analysis with LMI-based methods for real parametric uncertainty via parameter-dependent Lyapunov functions. *IEEE Trans. Autom. Control*, 46(4), 624–630.
- Peres, P.L.D., Geromel, J.C., and Souza, S.R. (1994). \mathcal{H}_2 output feedback control for discrete-time systems. In *Proc. 1994 Amer. Control Conf.*, volume 3, 2429–2433. Baltimore, USA.
- Richard, J.P. (2003). Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10), 1667–1694.
- Rugh, W.J. and Shamma, J.S. (2000). Research on gain scheduling. *Automatica*, 36(10), 1401–1425.
- Silva, J.V.V., Leite, V.J.S., and Silva, L.F.P. (2016). Controle de sistemas discretos no tempo com saturação de atuadores e atraso nos estados. In *XXI CBA*. Vitória, ES, Brazil.
- Souza, C., Leite, V.J.S., Silva, L.F.P., and Rubio Scola, I.E.J. (2017). Projeto de compensador anti-windup para sistemas sujeitos a atrasos nos estados com taxa de variação limitada. In *XIII SBAI*, 882–887. Porto Alegre, RS, Brazil.
- Verriest, E. (2010). Well-posedness of problems involving time-varying delays. In *Proc. of the 2010 Int. Symp. on the Mathematical Theory of Networks and Systems*. Budapest, Hungary.
- Wu, L., Shi, P., Wang, C., and Gao, H. (2006). Delay-dependent robust \mathcal{H}_∞ and L_2 -to- L_∞ filtering for LPV systems with both discrete and distributed delays. *IEE Proc. — Control Theory & Appl.*, 153(4), 483–492.
- Zhang, L., Shi, P., and Boukas, E.K. (2007). \mathcal{H}_∞ output-feedback control for switched linear discrete-time systems with time-varying delays. *Int. J. Control*, 80(8), 1354–1365.
- Zhang, X., Tsiotras, P., and Knospe, C. (2002). Stability analysis of LPV time-delayed systems. *Int. J. Control*, 75(7), 538–558.
- Zope, R., Mohammadpour, J., Grigoriadis, K., and Franchek, M. (2012). Delay-dependent output feedback control of time-delay LPV systems. In J. Mohammadpour and C.W. Scherer (eds.), *Control of Linear Parameter Varying Systems with Applications*, 279–299. Springer, New York.
- Zope, R., Mohammadpour, J., Grigoriadis, K., and Franchek, M. (2010). Robust fueling strategy for an SI engine modeled as an linear parameter varying time-delayed system. In *Proc. 2010 Amer. Control Conf.*, 4634–4639. Baltimore, MD, USA.